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A NEW HOMOTOPY PERTURBATION METHOD FOR SOLVING POISSON EQUATION

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ABSTRACT
In this paper, we applies a new homotopy perturbation method (NHPM),to find the exact solution of Poisson equation with Dirichelet and Neumann boundary conditions, to illustrate the ability and reliability of the method two examples are provided the results reveal that the method is very effective and simple.

KEYWORDS:.

INTRODUCTION
The new homotopy perturbation method (NHPM) was proposed by Biazar an Eslami [1] for solving two dimensional wave equation , The two most important steps in application of new homotopy perturbation method to construct a suitable homotopy equation and choose a suitable initial guess, Considerable research works have been conducted recently in applying this method to a class of linear and non-linear equations[2-5], The aim of this paper is to employ NHPM to obtain the exact solution of two Poisson equations, one with the Dirichlet boundary conditions and one with the Neumann boundary conditions., the difference between (NHPM) and standard (HPM) [6-10] is starts from the form of initial approximation of the solution. J. Biazar et all [11], and Selcuk Yıldırım [12] are obtained the solution of Poisson equation by HPM .in this letter, the basic idea of (NHPM) is given in section (2), we obtain the exact solution of Poisson equation in section (3) the last section (4) is reserved for conclusion.

BASIC IDEAS OF THE METHOD
Consider the two dimensional Poisson equation

$$u_{xx} + u_{yy} = f(x,y) \qquad 0 < x < a, 0 < y < b \tag{1}$$

Subject to the boundary condition

$$\begin{cases} u(0,y) = 0, \; u(a,y) = f(y), \\ u(x,0) = 0, \; u(x,b) = 0. \end{cases}$$

To solve Eq. (1) we construct the following homotopy

$$(1-p)\left(\frac{\partial^2 v}{\partial x^2} - u_0\right) + p\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - f(x,y)\right) = 0$$

Or:

$$\frac{\partial^2 v}{\partial x^2} = u_0 - p\left(u_0 + \frac{\partial^2 v}{\partial y^2} - f(x,y)\right) \tag{2}$$

Applying the inverse operator $L^{-1} = \int_0^x \int_0^x (\circ)\, dx dx$ to both sides of Eq. (2), we obtain

$$v(x,y) = v(0,y) + xv_x(0,y) + \int_0^x\int_0^x u_0\,dxdx - p\int_0^x\int_0^x\left(u_0 + \frac{\partial^2 v}{\partial y^2} - f(x,y)\right)dxdx \tag{3}$$

Where $v(0,y) = u(0,y)$ and $v_x(0,y) = u_x(0,y)$, suppose $u_x(0,y) = g(y)$

Assume the solution of Eq. (3) in the following form:

$$v = v_0 + pv_1 + p^2 v_2 + \dots \tag{4}$$

Suppose the initial approximation of the solution $u_0(x,y)$, is in the form

$$u_0(x,y) = \sum_{n=0}^{\infty} a_n(y) p_n(x) \tag{5}$$

Where $a_0(y), a_1(y), a_2(y), \dots$ are unknown coefficients and $q_0(y), q_1(y), q_2(y), \dots$ Are specified functions depending on the problem. Substituting (4) and (5) in to Eq. (3) and comparing coefficients of terms with identical powers p, we get

$$p^0 : v_0(x,y) = xg(y) + \int_0^x\int_0^x u_0\,dxdx$$

$$p^1 : v_1(x,y) = -\int_0^x\int_0^x\left(u_0 + \frac{\partial^2 v_0}{\partial y^2} - f(x,y)\right)dxdx \tag{6}$$

$$p^2 : v_2(x,y) = -\int_0^x\int_0^x \frac{\partial^2 v_1}{\partial y^2}\,dxdx$$

$$\vdots$$

Considering the hypothesis $v_1(x,y) = 0$ then the result in: $v_2(x,y) = v_3(x,y) = \dots = 0$

Therefore the exact solution would be obtained as the following $v(x,y) = v_0(x,y)$

It is important to note that if $u_0(x,y)$ is analytic at $x = x_0$ then its Taylor series is defined as $u_0(x,y) = \sum_{n=0}^{\infty} a_n(y)(x - x_0)^n$ which can be used in Eq. (6)

APPLICATION OF (NHPM):

Example 1. Consider the two dimensional Poisson equation in the form:

$$u_{xx} + u_{yy} = xy \qquad 0 < x, y < \pi, \tag{7}$$

Subject to the boundary condition:

$$\begin{cases} u(0,y) = 0,\ u(\pi,y) = \frac{1}{6}\pi^3 y, \\ u(x,0) = 0,\ \ u(x,\pi) = \frac{1}{6}\pi x^3 + \sin x \sinh \pi. \end{cases}$$

In order to solve Eq. (7) using (NHPM), we construct the following homotopy:

$$\frac{\partial^2 v}{\partial x^2} = u_0 - p\left(u_0 + \frac{\partial^2 v}{\partial y^2} - xy\right) \tag{8}$$

Appling the inverse operator $L^{-1} = \int_0^x\int_0^x (\circ)\,dxdx$ to both sides of Eq. (8) we obtain

$$v(x,y) = v(0,y) + xv_x(0,y) + \int_0^x\int_0^x u_0 dxdx - p\int_0^x\int_0^x \left(u_0 + \frac{\partial^2 v}{\partial y^2} - xy \right) dxdx \qquad (9)$$

Substituting v from Eq. (4) in to Eq. (9) an equating the term with like power p we can obtain

$$p^0 : v_0(x,y) = v(0,y) + xv_x(0,y) + \int_0^x\int_0^x u_0 dxdx$$

$$p^1 : v_1(x,y) = -\int_0^x\int_0^x \left(u_0 + \frac{\partial^2 v_0}{\partial y^2} - xy \right) dxdx \qquad (10)$$

$$p^2 : v_2(x,y) = -\int_0^x\int_0^x \frac{\partial^2 v_1}{\partial y^2} dxdx$$

$$\vdots$$

Suppose $u_0(x,y) = \sum_{n=0}^{\infty} a_n x^n$, $v(0,y) = u(0,y), v_x(0,y) = u_x(0,y) = f(y)$ then we have:

$$v_1(x,y) = \left(-\frac{a_0(y)}{2} \right) x^2 + \left(-\frac{a_1(y)}{6} - \frac{f''(y)}{6} + \frac{y}{6} \right) x^3 + \left(-\frac{a_2(y)}{12} - \frac{a_0''(y)}{24} \right) x^4$$

$$+ \left(-\frac{a_3(y)}{20} - \frac{a_1''(y)}{120} \right) x^5 + \ldots = 0$$

It can be easily shown that:

$$a_0(y) = 0 \ , \quad a_1(y) = y - f''(y) \ , \quad a_2(y) = 0 \quad , \ a_3(y) = \frac{f^{(4)}(y)}{6}, \ldots$$

This implies that

$$u(x,y) = v_0(x,y) = xf(y) - \frac{x^3}{3!}(f''(y) - y) + \frac{x^5}{5!} f^{(4)}(y) + \ldots$$

To determined the function $f(y)$, we use the inhomogeneous boundary condition $u(\pi,y) = \frac{1}{6}\pi x^3 + \sin x \sinh \pi$, we obtain $f(y) = \sinh y$ Therefore, the exact solution of Eq. (10) becomes as :

$$u(x,y) = \frac{1}{6}x^3 y + \sin x \sinh y$$

Example 2. Consider the two dimensional Poisson equation

$$u_{xx} + u_{yy} = xy \qquad 0 < x, y < \pi, \qquad (11)$$

Subject to the boundary condition:

$$\begin{cases} u_x(0,y) = \frac{1}{6}y^3, \ u_x(\pi,y) = \frac{1}{6}\pi^3, \\ u_y(x,0) = \cos x, \ \ u_y(x,\pi) = \frac{1}{2}\pi^2 x + \cos x \cosh \pi. \end{cases}$$

To solve Eq. (11) by (NHPM), we construct the following homotopy

$$\frac{\partial^2 v}{\partial x^2} = u_0 - p\left(u_0 + \frac{\partial^2 v}{\partial y^2} - xy \right) \qquad (12)$$

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By integration of Eq. (12), we have:

$$v(x,y)=v(0,y)+xv_x(0,y)+\int_0^x\int_0^x u_0\,dxdx-p\int_0^x\int_0^x\left(u_0+\frac{\partial^2 v}{\partial y^2}-xy\right)dxdx \qquad (13)$$

Assume the solution of Eq. (13) in the form (4) substituting (4) into Eq. (13) and equating terms with like powers of p, results in

$$\begin{aligned} p^0 &: v_0(x,y)=v(0,y)+xv_x(0,y)+\int_0^x\int_0^x u_0\,dxdx \\ p^1 &: v_1(x,y)=-\int_0^x\int_0^x\left(u_0+\frac{\partial^2 v_0}{\partial y^2}-xy\right)dxdx \\ p^2 &: v_2(x,y)=-\int_0^x\int_0^x \frac{\partial^2 v_1}{\partial y^2}\,dxdx \\ &\vdots \end{aligned} \qquad (14)$$

Suppose $u_0(x,y)=\sum_{n=0}^{\infty} a_n x^n$ $,v(0,y)=u(0,y)=f(y),v_x(0,y)=u_x(0,y)$ then we have:

$$v_1(x,y)=\left(-\frac{a_0(y)}{2}-\frac{f''(y)}{2}\right)x^2+\left(-\frac{a_1(y)}{6}\right)x^3+\left(-\frac{a_0''(y)}{24}-\frac{a_2(y)}{12}\right)x^4+$$

$$\left(-\frac{a_3(y)}{20}-\frac{a_1''(y)}{120}\right)x^4+\ldots=0$$

It can be easily shown that:

$$a_0(y)=-f''(y)\,,\quad a_1(y)=0,\quad a_2(y)=\frac{f^{(4)}(y)}{2}\ ,\ a_3(y)=0\,,\ldots$$

This implies that

$$u(x,y)=v_0(x,y)=f(y)+\frac{xy^3}{6}-\frac{x^2}{2!}f''(y)+\frac{x^5}{4!}f^{(4)}(y)+\ldots$$

To determined the function $f(y)$, we use the inhomogeneous boundary condition $u_y(x,\pi)=\frac{1}{2}\pi^2 x+\cos x\cosh\pi$, we obtain $f(y)=\cosh y$, so the exact solution is:

$$u(x,y)=\frac{1}{6}xy^3+\cosh y\left(1-\frac{x^2}{2!}+\frac{x^4}{4}+\ldots\right)$$

It important to note that the Neumann problem that the solution determined
up to an Additive constant cannot be determined by (NHPM), the solution should be

$$u(x,y)=\frac{1}{6}xy^3+\cos x\cosh y+c$$

Conclusions

In this article, a new homotopy perturbation method (NHPM) has been successfully applied to obtain the exact solution of Poisson equation with the Dirichlet and Neumann boundary conditions, in this method the first approximate solution has been use to reach the exact solution of the problem, that result reveal the method explicit, effective and easy to use.

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